Exercise 36

Find y'' by implicit differentiation.

$$x^2 + xy + y^2 = 3$$

Solution

Differentiate both sides with respect to x.

$$\frac{d}{dx}(x^2 + xy + y^2) = \frac{d}{dx}(3)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = 0$$

$$(2x) + \left[\frac{d}{dx}(x)\right] y + x\left[\frac{d}{dx}(y)\right] + (2y) \cdot \frac{d}{dx}(y) = 0$$

$$2x + (1)y + x(y') + 2yy' = 0$$

Solve for y'.

$$(x + 2y)y' = -2x - y \tag{1}$$

$$y' = -\frac{2x + y}{x + 2y}$$

Differentiate both sides of equation (1) with respect to x to get y''.

$$\frac{d}{dx}[(x + 2y)y'] = \frac{d}{dx}(-2x - y)$$

$$\left[\frac{d}{dx}(x + 2y)\right] y' + (x + 2y)\left[\frac{d}{dx}(y')\right] = -2 - y'$$

$$(1 + 2y')y' + (x + 2y)(y'') = -2 - y'$$

Bring the terms with y' to the right side.

$$\begin{aligned}
(x + 2y)y'' &= -2 - 2y' - 2(y')^2 \\
&= -2 - 2\left(-\frac{2x + y}{x + 2y}\right) - 2\left(-\frac{2x + y}{x + 2y}\right)^2 \\
&= -\frac{2(x + 2y)^2}{(x + 2y)^2} + \frac{2(2x + y)(x + 2y)}{(x + 2y)^2} - 2\frac{(2x + y)^2}{(x + 2y)^2} \\
&= \frac{-2(x + 2y)^2 + 2(2x + y)(x + 2y) - 2(2x + y)^2}{(x + 2y)^2} \\
&= \frac{-6x^2 - 6xy - 6y^2}{(x + 2y)^2}
\end{aligned}$$

Now solve for y''.

$$\begin{aligned} y'' &= \frac{-6(x^2 + xy + y^2)}{(x + 2y)^3} \\ &= \frac{-6(3)}{(x + 2y)^3} \\ &= -\frac{18}{(x + 2y)^3} \end{aligned}$$